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Grades 9–10

Sample Math Practice

QUESTIONS

Easy Questions (1–7)

1. A bank requires 4-digit PIN codes to contain at least one even digit AND at least one odd digit. How many valid PIN codes are possible?

(A) 8125 (B) 8750 (C) 9375 (D) 9500 (E) 9750

2. Herr Waxi buys 100 candles. Each day he burns down one candle. From the leftovers of seven burned candles, he can always make a new candle. After how many days does he have to buy new candles?

(A) 112 (B) 114 (C) 115 (D) 116 (E) 117

3. Tom and Mary play a game with a coin. When the coin shows heads, Mary wins and Tom must give her two sweets. When the coin shows tails, Tom wins and Mary must give him three sweets. After 30 throws of the coin they each have the same number of sweets as they had at the start. How often has Tom won?

(A) 6 (B) 12 (C) 18 (D) 24 (E) 30

4. Sven writes five different single-digit positive whole numbers on a board. He realizes that no sum of two of these numbers equals 10. Which of the following numbers has Sven definitely written on the board?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

5. Each inhabitant of a distant planet has at least two ears. Three inhabitants named Imi, Dimi, and Trimi meet. Imi says: "I can see 8 ears." Dimi replies: "I can see 7 ears." Trimi says: "Strange, I can only see 5 ears." None of them can see their own ears. How many ears does Trimi have?

(A) 2 (B) 4 (C) 5 (D) 6 (E) 7

6. In the sequence $2, 6, 18, 54, \ldots$, each term is three times the previous term. What is the units digit of the 20th term?

(A) 2 (B) 4 (C) 6 (D) 8 (E) 0

7. A positive integer is called "special" if it equals twice the sum of its digits. How many two-digit special numbers are there?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Medium Questions (8–14)

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8. A teacher writes fractions on the board: $\frac{1}{1\times 2}, \frac{1}{2\times 3}, \frac{1}{3\times 4}, \ldots, \frac{1}{99\times 100}$. She asks her students to find the sum. What is it?

(A) $\frac{99}{100}$ (B) $\frac{49}{100}$ (C) $\frac{1}{100}$ (D) $\frac{50}{101}$ (E) $\frac{98}{99}$

9. A frog starts at position 0 on a number line. Each jump is either $+5$ or -3. What is the minimum number of jumps needed to reach position 1?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

10. Jean-Philippe has n^3 small cubes of equal size. He assembles them into one large $n \times n \times n$ cube and paints all outer surfaces. The number of small cubes with exactly one painted face equals the number of small cubes with no painted faces. What is n?

(A) 4 (B) 6 (C) 7 (D) 8 (E) 10

11. A gardener plants flowers in a row. She uses red (R) and yellow (Y) flowers, placing them so that no three consecutive flowers are the same color. If she plants exactly 10 flowers starting with R, how many valid arrangements are possible?

(A) 55 (B) 89 (C) 109 (D) 144 (E) 178

12. Maya notices that if she adds 2025 to her age and divides by her age, she gets an integer. If Maya is a teenager (13-19 years old), how old is she?

(A) 13 (B) 15 (C) 17 (D) 18 (E) 19

13. Compute the sum: $\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \frac{1}{7\times 9} + \frac{1}{9\times 11}$.

(A) $\frac{4}{11}$ (B) $\frac{5}{11}$ (C) $\frac{5}{22}$ (D) $\frac{6}{11}$ (E) $\frac{1}{2}$

14. A librarian arranges books on shelves. She notices that when she places books in rows of 7, there are 3 left over. When she places them in rows of 11, there are also 3 left over. What is the smallest number of books she could have (greater than 50)?

(A) 52 (B) 59 (C) 66 (D) 71 (E) 80

Hard Questions (15–21)

15. How many integers k have the property that $k + 6$ is a multiple of $k - 6$?

(A) 0 (B) 4 (C) 6 (D) 8 (E) 12

16. The digit sum of N is three times the digit sum of $N+1$. What is the smallest possible digit sum of N?

(A) 3 (B) 9 (C) 12 (D) 15 (E) 27

17. Evaluate: $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots+\frac{1}{18\cdot 19\cdot 20}$

(A) $\frac{95}{380}$ (B) $\frac{189}{760}$ (C) $\frac{94}{380}$ (D) $\frac{1}{20}$ (E) $\frac{19}{380}$

18. What is the smallest positive integer that leaves remainder 1 when divided by 2, remainder 2 when divided by 3, remainder 3 when divided by 4, and remainder 4 when divided by 5?

(A) 29 (B) 49 (C) 59 (D) 119 (E) 179

19. What is the remainder when $1! + 2! + 3! + \cdots + 100!$ is divided by 15?

(A) 0 (B) 3 (C) 6 (D) 9 (E) 12

20. If $p(x) = x^4 + ax^3 + bx^2 + cx + d$ satisfies $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, and $p(4) = 4$, what is $p(5)$?

(A) 5 (B) 25 (C) 29 (D) 125 (E) 149

21. For $x > 0$, define $f(x) = x + \frac{9}{x}$. What is the minimum value of $f(x)$?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 9

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SOLUTIONS

Easy Questions (1–7)

1. Your first instinct might be to count directly: how many PINs have exactly one even digit? Exactly two? But that is a nightmare of cases.

Flip it around. What PINs do NOT work? Only two types fail:

All digits even (only 0, 2, 4, 6, 8): $5^4 = 625$ possibilities.

All digits odd (only 1, 3, 5, 7, 9): $5^4 = 625$ possibilities.

Total PINs: $10^4 = 10{,}000$

Valid PINs: $10{,}000 - 625 - 625 = 8750$

This technique is called **complement counting**: instead of counting what you want, count what you do not want and subtract.

Answer: (B) 8750

2. This problem requires tracking leftovers through multiple rounds of recycling.

Round 1: Start with 100 candles. Burn them all over 100 days. Collect 100 leftovers.

Round 2: From 100 leftovers, make $\lfloor 100/7 \rfloor = 14$ new candles with $100 - 98 = 2$ leftovers remaining. Burn 14 candles over 14 days. Now have $14 + 2 = 16$ leftovers.

Round 3: From 16 leftovers, make $\lfloor 16/7 \rfloor = 2$ new candles with $16 - 14 = 2$ leftovers remaining. Burn 2 candles over 2 days. Now have $2 + 2 = 4$ leftovers.

Round 4: 4 leftovers cannot make another candle (need 7).

Total days: $100 + 14 + 2 = 116$.

The insight: Each burned candle contributes $1/7$ of a future candle. The total "candle value" is $100 + 100/7 + 100/49 + \ldots$, but we must track the discrete leftovers carefully.

Answer: (D) 116

3. This is a **system of linear equations** problem in disguise.

Let Tom win t times and Mary win m times. Total: $t + m = 30$.

When Tom wins, he gains 3 sweets. When Mary wins, she gains 2 sweets from Tom.

Net change for Tom: $3t - 2m = 0$ (they end up with the same as they started).

So $3t = 2m$. Now we have two equations:

- $t + m = 30$
- $3t = 2m$

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From the first equation: $m = 30 - t$.

Substitute into the second: $3t = 2(30 - t) = 60 - 2t$

$5t = 60$, so $t = 12$.

Answer: (B) 12

4. Five different single-digit positive numbers (1-9) with no two summing to 10.

Pairs summing to 10: $(1, 9), (2, 8), (3, 7), (4, 6)$.

From each pair, we can use at most one number.

That gives us at most 4 numbers from these pairs. We need 5 numbers.

The fifth must be 5 (the only number not in any pair summing to 10).

So 5 is definitely on the board.

The key insight: This is a **pigeonhole argument**. The pairs partition $\{1, 2, 3, 4, 6, 7, 8, 9\}$ into 4 groups. Picking 5 numbers forces us to include the "loner" number 5.

Answer: (E) 5

5. Let Imi have i ears, Dimi have d ears, and Trimi have t ears.

Imi sees Dimi and Trimi: $d + t = 8$.

Dimi sees Imi and Trimi: $i + t = 7$.

Trimi sees Imi and Dimi: $i + d = 5$.

The trick: Add all three equations:

$2(i + d + t) = 8 + 7 + 5 = 20$

So $i + d + t = 10$.

Now subtract each equation from this total:

From $d + t = 8$: $i = 10 - 8 = 2$.

From $i + t = 7$: $d = 10 - 7 = 3$.

From $i + d = 5$: $t = 10 - 5 = 5$.

Trimi has 5 ears.

Answer: (C) 5

6. The sequence is $2, 6, 18, 54, \ldots$ where each term is $\times 3$.

Wrong approach: Computing 2×3^{19} directly. The number is astronomically large.

Better approach: We only care about the units digit, so track how it cycles.

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Find the pattern in units digits:

- Term 1: 2
- Term 2: $2 \times 3 = 6$
- Term 3: $6 \times 3 = 18 \to$ units digit 8
- Term 4: $8 \times 3 = 24 \to$ units digit 4
- Term 5: $4 \times 3 = 12 \to$ units digit 2 (back to start!)

The units digits cycle: $2, 6, 8, 4, 2, 6, 8, 4, \ldots$ with **period 4**.

Finding the 20th term's position: $20 \div 4 = 5$ remainder 0.

Remainder 0 means the 20th term is at the same position as the 4th term in the cycle.

Units digit of 20th term: 4.

Verification: Term 4 has units digit 4 (from 54). Term 8 has units digit 4 (from $2 \times 3^7 = 4374$). The pattern holds.

The insight: When only the units digit matters, compute mod 10. Patterns always emerge because there are only 10 possible units digits.

Answer: (B) 4

7. Let the two-digit number be $\overline{ab} = 10a + b$, where a is the tens digit ($1 \leq a \leq 9$) and b is the units digit ($0 \leq b \leq 9$).

The condition: $10a + b = 2(a + b)$

Expand and simplify:

$$10a + b = 2a + 2b$$

$$8a = b$$

Key constraint: Since b must be a single digit ($0 \leq b \leq 9$) and $8a = b$:

- If $a = 1$: $b = 8$. The number is 18.
- If $a = 2$: $b = 16$. Not a digit!
- If $a \geq 2$: $b \geq 16$, always too large.

Verification: Is 18 special? Digit sum $= 1 + 8 = 9$. Twice the digit sum $= 18$. Yes!

The insight: The place value of the tens digit (worth 10) versus its contribution to the digit sum (worth 1) creates a severe constraint. The "excess value" of 8 per tens digit must be balanced by the units digit alone.

Only one special two-digit number exists: 18.

Answer: (B) 1

Medium Questions (8–14)

8. $\displaystyle\sum_{k=1}^{99} \frac{1}{k(k+1)}$

Use **partial fractions**: $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$

Why does this work? Multiply the right side out:

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}$$

This is a **telescoping sum**:

$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right)$

The $-\frac{1}{2}$ from the first term cancels with the $+\frac{1}{2}$ from the second. Everything in the middle vanishes!

What survives? Only the first and last terms: $1 - \dfrac{1}{100} = \dfrac{99}{100}$

Answer: (A) $\dfrac{99}{100}$

9. Start at 0, reach 1 using jumps of $+5$ or -3.

Let a = number of $+5$ jumps, b = number of -3 jumps.

We need: $5a - 3b = 1$.

This is a **linear Diophantine equation**. We want the solution with minimum $a + b$.

Solve for small values:

$a = 2, b = 3$: $10 - 9 = 1$. Total jumps: 5.

Can we do better?

$a = 1$: $5 - 3b = 1 \Rightarrow b = 4/3$. Not an integer.

$a = 0$: $-3b = 1$. Impossible.

Minimum jumps: 5 (two $+5$ and three -3).

Verification: $+5 + 5 - 3 - 3 - 3 = 10 - 9 = 1$. Works!

Answer: (C) 5

10. An $n \times n \times n$ cube made of unit cubes.

Cubes with exactly 1 painted face: These are the face centers (not on edges). Each face has $(n-2)^2$ such cubes. Total: $6(n-2)^2$.

Cubes with no painted faces: These form the interior cube of size $(n-2)^3$.

Set them equal: $6(n-2)^2 = (n-2)^3$

If $n > 2$, divide by $(n-2)^2$: $6 = n - 2$, so $n = 8$.

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Verification: For $n = 8$:

- One-face cubes: $6 \times 36 = 216$
- Interior cubes: $6^3 = 216$

They match!

Answer: (D) 8

11. Count arrangements of 10 flowers starting with R, where no three consecutive flowers are the same color.

State machine approach: Track how the sequence ends:

- $s_1(n)$ = sequences ending in single R (previous was Y)
- $s_2(n)$ = sequences ending in RR
- $s_3(n)$ = sequences ending in single Y (previous was R)
- $s_4(n)$ = sequences ending in YY

Transitions:

- From single R: add R $\rightarrow$ RR, or add Y $\rightarrow$ single Y
- From RR: cannot add R (would make RRR), add Y $\rightarrow$ single Y
- From single Y: add R $\rightarrow$ single R, or add Y $\rightarrow$ YY
- From YY: cannot add Y, add R $\rightarrow$ single R

Initial condition ($n = 1$): $(s_1, s_2, s_3, s_4) = (1, 0, 0, 0)$

Computing step by step, the totals follow a Fibonacci-like pattern: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89$.

The 10th term is 89.

Answer: (B) 89

12. $(2025 + \text{age})/\text{age}$ is an integer.

$$\frac{2025 + a}{a} = \frac{2025}{a} + 1$$

For this to be an integer, a must divide 2025.

$2025 = 81 \times 25 = 3^4 \times 5^2$

Divisors of 2025: 1, 3, 5, 9, 15, 25, 27, 45, 75, 81, 135, 225, 405, 675, 2025.

Teenage divisors (13-19): only 15.

Maya is 15 years old.

Verification: $(2025 + 15)/15 = 2040/15 = 136$. It is an integer!

Answer: (B) 15

13. Compute $\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \frac{1}{7 \times 9} + \frac{1}{9 \times 11}$.

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The trick is **partial fractions** with a twist. Each term has the form $\frac{1}{(2k-1)(2k+1)}$. The gap between factors is 2, so:

$$\frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$$

Now watch what happens when we add the terms:

$= \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)\right]$

Everything in the middle cancels! This is called **telescoping**.

What survives? Only $1 - \frac{1}{11} = \frac{10}{11}$.

Final answer: $\frac{1}{2} \times \frac{10}{11} = \frac{5}{11}$.

Answer: (B) $\frac{5}{11}$

14. The number of books B satisfies:

- $B \equiv 3 \pmod 7$
- $B \equiv 3 \pmod{11}$

Since the remainders are the same, we can write $B - 3$ is divisible by both 7 and 11.

Since $\gcd(7, 11) = 1$, we have $B - 3$ divisible by $7 \times 11 = 77$.

So $B = 77n + 3$ for some non-negative integer n.

Values: $3, 80, 157, \ldots$

The smallest value greater than 50 is $77 \times 1 + 3 = 80$.

Verification: $80 = 7 \times 11 + 3$ and $80 = 11 \times 7 + 3$. Both give remainder 3.

Answer: (E) 80

Hard Questions (15–21)

15. We need $(k-6) \mid (k+6)$.

Wrong approach: Testing random values of k to see which work. This is slow and you might miss negative values or zero.

Better approach: Rewrite $k + 6$ in terms of $k - 6$:

$$k + 6 = (k - 6) + 12$$

If $(k-6)$ divides $(k+6)$, it must also divide 12 (since it already divides $k-6$).

So $(k-6)$ must be a divisor of 12.

Divisors of 12: $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$. Count: 12 divisors.

For each divisor d, we get $k = d + 6$:

$$k \in \{7, 8, 9, 10, 12, 18, 5, 4, 3, 2, 0, -6\}$$

All 12 values work.

Answer: (E) 12

16. Let $S(N)$ denote the digit sum of N. We need $S(N) = 3 \cdot S(N+1)$.

Key observation: When adding 1 to a number, the digit sum usually increases by 1. But when there is a carry (like $99 \to 100$), the digit sum drops significantly.

The digit sum drops when N ends in 9s. For example:

- $N = 99$: $S(N) = 18$, $S(100) = 1$. Ratio: 18.
- $N = 999$: $S(N) = 27$, $S(1000) = 1$. Ratio: 27.

We need ratio exactly 3. Try N ending in some 9s:

$N = 29$: $S(29) = 11$, $S(30) = 3$. Ratio $= 11/3$. Not integer.

$N = 39$: $S(39) = 12$, $S(40) = 4$. Ratio $= 3$. **Yes!**

So the smallest digit sum of N is $S(39) = 12$.

Why this works: When $N = a9$ (two digits ending in 9), $S(N) = a + 9$ and $S(N+1) = a + 1$. Ratio $= (a+9)/(a+1) = 3$ gives $a + 9 = 3a + 3$, so $a = 3$.

Answer: (C) 12

17. $\displaystyle\sum_{k=1}^{18} \frac{1}{k(k+1)(k+2)}$

Wrong approach: Computing each fraction and finding a common denominator. With 18 terms and denominators like $18 \times 19 \times 20 = 6840$, this becomes a nightmare.

Better approach: Use **partial fractions**. The key identity:

$$\frac{1}{k(k+1)(k+2)} = \frac{1}{2}\left(\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right)$$

Why? This comes from the general pattern for triple products. Verify by finding a common denominator.

This is a **telescoping sum**!

$$\frac{1}{2}\left[\left(\frac{1}{1\cdot2} - \frac{1}{2\cdot3}\right) + \left(\frac{1}{2\cdot3} - \frac{1}{3\cdot4}\right) + \cdots + \left(\frac{1}{18\cdot19} - \frac{1}{19\cdot20}\right)\right]$$

Most terms cancel, leaving only the first and last:

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$$= \frac{1}{2}\left(\frac{1}{1\cdot 2} - \frac{1}{19\cdot 20}\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{380}\right)$$

$$= \frac{1}{2}\left(\frac{190-1}{380}\right) = \frac{1}{2}\cdot\frac{189}{380} = \frac{189}{760}$$

Answer: (B) $\frac{189}{760}$

18. The conditions are:

- $n \equiv 1 \pmod 2$ (remainder 1 when divided by 2)
- $n \equiv 2 \pmod 3$ (remainder 2 when divided by 3)
- $n \equiv 3 \pmod 4$ (remainder 3 when divided by 4)
- $n \equiv 4 \pmod 5$ (remainder 4 when divided by 5)

Key observation: Notice the pattern! Each remainder is one less than the divisor:

- $n \equiv -1 \pmod 2$
- $n \equiv -1 \pmod 3$
- $n \equiv -1 \pmod 4$
- $n \equiv -1 \pmod 5$

So $n + 1$ is divisible by 2, 3, 4, and 5.

$\text{lcm}(2, 3, 4, 5) = 60$.

So $n + 1 = 60k$ for some positive integer k, meaning $n = 60k - 1$.

Smallest positive: $n = 59$.

Verification: $59 = 2 \times 29 + 1$, $59 = 3 \times 19 + 2$, $59 = 4 \times 14 + 3$, $59 = 5 \times 11 + 4$. All correct!

Answer: (C) 59

19. $1! + 2! + 3! + \cdots + 100! \pmod{15}$

Wrong approach: Trying to compute 100! or even thinking about large factorials. The numbers grow astronomically fast.

Key insight: For $n \geq 5$, we have $n! = n \times (n-1) \times \cdots \times 5 \times \cdots$, which contains both factors 3 and 5.

So $n! \equiv 0 \pmod{15}$ for all $n \geq 5$.

Only $1! + 2! + 3! + 4!$ contribute to the remainder:

$1! = 1$

$2! = 2$

$3! = 6$

$4! = 24$

Sum $= 1 + 2 + 6 + 24 = 33$.

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$33 \mod 15 = 33 - 2 \times 15 = 33 - 30 = 3.$

Answer: (B) 3

20. $p(x) = x^4 + ax^3 + bx^2 + cx + d$ with $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 4$.

Elegant approach: Define $q(x) = p(x) - x$.

Then $q(1) = q(2) = q(3) = q(4) = 0$.

So $q(x)$ is a degree-4 polynomial with roots 1, 2, 3, 4:

$$q(x) = (x - 1)(x - 2)(x - 3)(x - 4)$$

(The leading coefficient is 1 since $p(x)$ has leading coefficient 1.)

Therefore: $p(x) = (x - 1)(x - 2)(x - 3)(x - 4) + x$

Computing $p(5)$:

$$p(5) = (5 - 1)(5 - 2)(5 - 3)(5 - 4) + 5 = 4 \times 3 \times 2 \times 1 + 5 = 24 + 5 = 29$$

The insight: Instead of solving a system of 4 equations in 4 unknowns, we recognized that $p(x) - x$ vanishes at four points, so it must be the unique monic degree-4 polynomial with those roots.

Answer: (C) 29

21. $f(x) = x + \frac{9}{x}$ for $x > 0$.

Using the AM-GM Inequality:

For positive numbers a and b: $\frac{a + b}{2} \geq \sqrt{ab}$, with equality when $a = b$.

Apply to x and $\frac{9}{x}$:

$$\frac{x + \frac{9}{x}}{2} \geq \sqrt{x \cdot \frac{9}{x}} = \sqrt{9} = 3$$

So: $x + \frac{9}{x} \geq 6$

Equality condition: $x = \frac{9}{x}$, which gives $x^2 = 9$, so $x = 3$ (since $x > 0$).

Verification: $f(3) = 3 + \frac{9}{3} = 3 + 3 = 6$. The minimum is indeed 6.

Why AM-GM is powerful: This inequality appears constantly in optimization problems. The pattern $x + \frac{k}{x}$ always has minimum $2\sqrt{k}$, achieved at $x = \sqrt{k}$.

Answer: (C) 6

End of Free Sample

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